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SOLUTIONS OF PROBLEMS IN NUMBER SIX, VOL. VI.

SOLUTIONS of problems in No. 6, Vol. VI, have been received as follows;

From R. J. Adcock, 284, 286; Prof. W. P. Casey, 283; George Eastwood, 282, 283, 287; Prof. E. W. Hyde, 283, 287; George H. Harvill, 287; O. L. Mathiot, 283; Prof. E. B. Seitz, 283, 284, 285, 286, 287.

Prof. Casey sent solutions of 276, 277 and 280, in time for the notice in last No., but they were accidentally overlooked; and W. E. Heal sent a good algebraic solution of 277 which was received too late for the notice in last No.

281. No solution has been received.

282. "Suppose a manufacturer of textile fabrics, who can operate sixty looms, ascertains by trial, (1) that when he makes his weavers sure of a dollar a day with constant work, then each loom, when it runs a whole day, will net three dollars; (2) that, from the nature of the work to be done, all the looms are never in running order at the same time; (3) that at any time it is equally possible that one, two, or three, or any number of them may be standing for repairs. Now, under these conditions, it is required to assign the least number of weavers to be placed under permanent pay, so that the average daily profits may be the greatest possible."

SOLUTION BY GEORGE EASTWOOD, SAXONVILLE, MASS.

Represent the number of weavers required by x and the average daily profits by y. Then,

If 1 loom is out of order, the profits are evidently $3x$,
" 2 l'ms are " " " " " " $3x$,
" $60-x$ " " " " " " " " $3x$,
" $60-x+1$ " " " " " " " " $3(x-1)-1$,
" $60-x+2$ " " " " " " " " $3(x-2)-2$,
" $60-1$ " " " " " " " " $3-(x-1)$,
" 60 " " " " " " " " $-x$.

The sum of these divided by 60 gives, for the daily average,

$$y = \tfrac{1}{60}[3(60-x)x + \tfrac{3}{2}(x-1)x - \tfrac{1}{2}(x+1)x]$$
$$= \tfrac{1}{60}(178x - 2x^2);$$
$$\therefore dy = \tfrac{1}{60}d.(178x - 2x^2) = 0,$$

which gives $x = \frac{1}{4}(178) = 44 + \frac{1}{2}$; hence the least number of weavers to be kept on permanent pay is 44, and the average daily profits of the manufacturer is \$66.008$\frac{1}{3}$.

Instead of 60 looms, let there be n looms, and let us suppose the profit on each loom, when it works a whole day, to be m times the pay of the weaver. Then, x and y being as before,

If 1 loom is not in working condition the profits are mx,
“ 2 looms are “ “ “ “ “ “ “ mx,
. . . .
“ $n-x$ “ “ “ “ “ “ “ “ “ mx,
“ $n-x+1$ “ “ “ “ “ “ “ “ “ $m(x-1)-1$,
“ $n-x+2$ “ “ “ “ “ “ “ “ “ $m(x-2)-2$,
. . . .
“ $n-1$ “ “ “ “ “ “ “ “ “ $m-(x-1)$,
“ n “ “ “ “ “ “ “ “ “ $-x$.

Summing and dividing by n, the average profit will be

$$y = \frac{1}{n}\left[m(n-x)x + \frac{m(x-1)x}{2} - \frac{(x+1)x}{2}\right]$$

$$= \frac{1}{n}\left[\left(mn - \frac{m+1}{2}\right)x - \frac{m+1}{2}x^2\right]$$

$$= \frac{m+1}{2n}\left[\left(\frac{mn}{m+1} - \frac{1}{2}\right)^2 - \left(x - \frac{mn}{m+1} + \frac{1}{2}\right)^2\right].$$

This is the greatest possible when

$$x = \frac{mn}{m+1} - \frac{1}{2},$$

or the integer nearest to this value of x. Hence the general value of y is

$$\frac{m+1}{2n}\left(\frac{mn}{m+1} - \frac{1}{2}\right)^2.$$

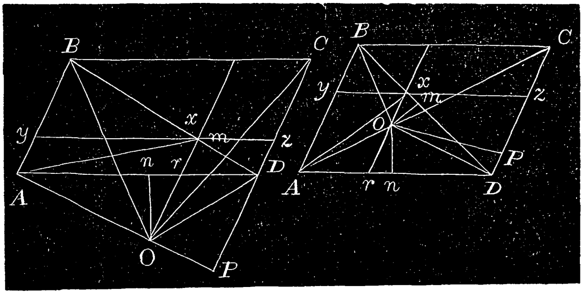
283. “If from any point in the plane of a parallelogram perpendiculars are let fall on the diagonal and on the two sides that contain this diagonal, the product of the diagonal by its perpendicular is equal to the sum of the products of the sides by their respective perpendiculars if the point falls outside of the parallelogram, or to their difference if it lies within the parallelogram.

Varignon’s Theorem: Mecanique Analytique, Tome I, p. 13.”

SOLUTION BY PROF. W. P. CASEY, SAN FRANCISCO, CAL.

Let $ABCD$ be the parallelogram and O the point; draw the perpendiculars Om, Op, On; and draw AO, BO, CO and DO. Through O draw Ox

parallel to AB, and through x draw yz parallel to AD. Now parallelog'm Cx = parallelogram Ax. To each add the parallelogram Dx, and ∴ par. Az = par. Cr, or triangle AxD = triangle DOC; but triangle $OBx = \triangle OAx$, to each add triangle DOx, and therefore $\triangle BOD$ = quad.$AxDO$ $= \triangle AxD \pm \triangle AOD = \triangle DOC \pm \triangle AOD$; that is, $BD \times Om = CD \times Op \pm AD \times On$. Q. E. D.



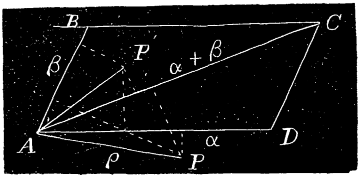
This theorem is of great use in the doctrine of statical moments.

SOLUTION (BY QUATERNIONS) BY PROF. E. W. HYDE, CIN., OHIO.

Let $AB = \beta$, $AD = \alpha$ and $AP = \rho$; then we have at once

$$V\alpha\rho + V\beta\rho = V(\alpha+\beta)\rho,$$

which contains the proof, since $U\alpha\rho = U\beta\rho = U(\alpha+\beta)\rho$, and therefore



$$T\alpha\rho + T\beta\rho = T(\alpha+\beta)\rho,$$

when P is outside the parallelogram; and

$$U\alpha\rho = -U\beta\rho = \pm U(\alpha+\beta)\rho,$$

and therefore

$$T\alpha\rho - T\beta\rho = \pm T(\alpha + \beta)\rho$$

when P is inside. These tensors are the products required.

SOLUTION BY O. L. MATHIOT, BALTIMORE, MARYLAND.

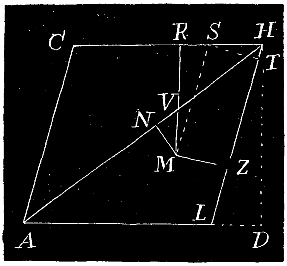
Referring to the marginal diagram, we find from the similar triangles ADH, MNV and HRV

$$MV = \frac{AH.MN}{AD},$$

$$RV = MR - MV = MR - \frac{AH.MN}{AD},$$

$$RH = \frac{AD.RV}{DH} = \frac{AD.RM - AH.MN}{DH}.$$

And from the similar triangles HDL, SHT we find



$$RS = (DL.RM) \div DH,$$

$$HS = HR - RS = \frac{AD.RM - AH.MN - DL.RM}{DH},$$

$$ST\,(=MZ)=\frac{DH.HS}{HL}=\frac{AD.RM-AH.MN-DL.RM}{HL}.$$

Substituting for AD its equivalent, $AL+DL$ we have

$$MZ=\frac{AL.RM+DL.RM-AH.MN-DL.RM}{HL},$$

$$\therefore MZ.HL=AL\,(=CH).RM-AH.MN,$$

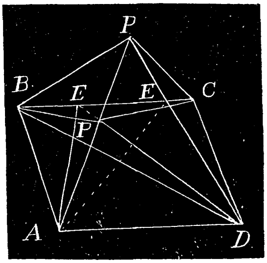
$$\text{or } AH.MN=CH.RM-HL.MZ;$$

which was to be proved.

In a similar manner the prop. may be proved for an external point.

[Prof. Hall writes, in relation to this theorem, "Lagrange says: 'Varignon shows, by a very simple construction, that in forming the triangles which have the diagonal and the two sides for bases, and the given point for a common summit, the triangle formed on the diagonal is, in the first case, equal to the sum, and, in the second case, to the difference of the two triangles formed on the sides; that which is in itself a beautiful theorem of geometry, independently of its application to mechanics.' "

At the risk of seeming tedious in the discussion of this proposition, we submit the following construction which presents a very direct and easy method of demonstration.



Let $ABCD$ represeat the parallelagram, and let P be either an external or an internal point; also, let E denote the intersection of DP, or DP produced, with the side BC. Then, because the sum of the areas of the triangles ABP and CDP is half the area of the parallelogram, we may easily prove that the triangles BCP and AEP have equal areas.

Now if from the quadrilateral $DABP$ we take the triangle ABD, which is half the parallelogram, we have left the triangle BDP, which has for its base the diagonal BD and for its summit the point P. And if from the same quadrilateral, $DABP$, we take the triang. $ADP \mp$ the triangle AEP (which difference or sum is half the parallelogram), we have left the triangles $ABP \pm AEP$, which sum or difference is therefore equal to the triangle BDP. But we have shown that the triangle AEP is equal to the triangl BCP; therefore the triangle BDP is equal to the triangle ABP ± the triangle BCP.—Ed.]

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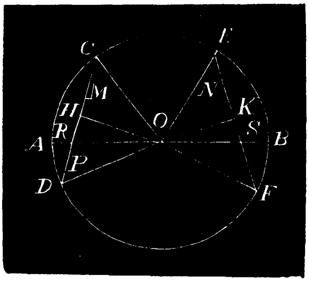
284. "*M*, *N* and *P*, *Q* are two pairs of random points within a circle, but on opposite sides of a given diameter AB: find the chance that the sum of the squares of the chords through M, P and N, Q is less than the square of the diameter."

SOLUTION BY PROF. E. B. SEITZ, KIRKSVILLE, MO.



Let CD and EF be the chords through M, P and N, Q, intersecting AB in R and S, and O the center of the circle. Draw OH and OK perpendicular to CD and EF.

Now the sum of the squares of CD and EF is less than the square of AB, if the sum of the angles COD and EOF is less than two right angles.

Let $OA = r$, $RM = w$, $RC = w'$, $RP = x$, $RD = x'$, $SN = y$, $SE = y'$, $SQ = z$, $SF = z'$, $\angle COH = \theta$, $\angle EOK = \varphi$, $\angle ROH = \psi$, and $\angle SOK = \mu$. Then we have

$$w' = r(\sin\theta + \cos\theta\tan\psi),$$
$$x' = r(\sin\theta - \cos\theta\tan\psi),$$
$$y' = r(\sin\varphi + \cos\varphi\tan\mu),$$
$$z' = r(\sin\varphi - \cos\varphi\tan\mu).$$

An element of surface at M is $r\sin\theta\, d\theta\, dw$; at P, $(w + x)\, d\psi\, dx$; at N, $r(\sin\varphi\, d\varphi\, dy$; and at Q, $(y+z)\, d\mu\, dz$. The limits of θ are 0 and $\frac{1}{2}\pi$; of φ, 0 and $\frac{1}{2}\pi-\theta$; of ψ, $-\theta$ and θ and doubled; of μ, $-\theta$ and θ, and doubled; of w, 0 and w'; of x, 0 and x'; of y, 0 and y'; and of z, 0 and z'.

Hence, since the whole number of ways the four points can be taken is $\frac{1}{16}\pi^4r^8$, the required chance is

$$p = \frac{64}{\pi^4r^8}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi-\theta}\int_{-\theta}^{\theta}\int_{-\phi}^{\phi}\int_0^{w'}\int_0^{x'}\int_0^{y'}\int_0^{z'} r\sin\theta\, d\theta\, r\sin\varphi\, d\varphi$$
$$\times\, d\psi\, d\mu\, dw\,(w+x)dx\, dy\,(y+z)dz$$
$$= \frac{64}{\pi^4}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi-\theta}\int_{-\theta}^{\theta}\int_{-\phi}^{\phi}(1-\cos^2\theta\sec^2\psi)(1-\cos^2\varphi\sec^2\mu)\sin^2\theta\sin^2\varphi$$
$$\times\, d\theta\, d\varphi\, d\psi\, d\mu$$
$$= \frac{256}{\pi^4}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi-\theta}(\theta-\sin\theta\cos\theta)(\varphi-\sin\varphi\cos\varphi)\sin^2\theta\sin^2\varphi\, d\theta\, d\varphi$$
$$= \frac{16}{\pi^4}\int_0^{\frac{1}{2}\pi}(\pi-2\theta-\sin 2\theta)^2(\theta-\sin\theta\cos\theta)\sin^2\theta\, d\theta$$
$$= \frac{1}{6}+\frac{2}{\pi^2}-\frac{320}{9\pi^4}.$$

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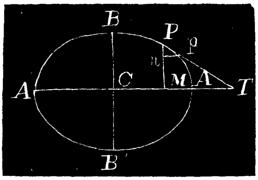
285. "If $E(e, x)$ denote the length of an elliptic arc, semi major axis unity and eccentricity e; prove, without the aid of the Integral Calculus, that

$$d\,E(e, x) = dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}."$$

SOLUTION BY PROFESSOR SEITZ.



Let $CA = a = 1$, $CB = b$, $CM = x$, $PM = y$, arc $BP = E(e, x)$, $Pp = dE(e, x)$, $np = dx$, $\angle PTM = Ppn = \theta$. Now by Analytical geometry we have

$$\tan\theta = \frac{b^2x}{a^2y} = \frac{x(1-e^2)}{y} = \frac{x\sqrt{(1-e^2)}}{\sqrt{(a^2-x^2)}} = \frac{x\sqrt{(1-e^2)}}{\sqrt{(1-x^2)}};$$

$$\therefore \sec\theta = \sqrt{(1+\tan^2\theta)} = \sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)},$$

$$\text{and } dE(e, x) = \sec\theta\, dx = dx\sqrt{\left(\frac{1-e^2x^2}{1-x^2}\right)}.$$

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286. "Three points are taken at random in a given circle, and a circle is drawn through them. What is the probability that the circle through the random points will be wholly in the given circle?"

SOLUTION BY R. J. ADCOCK, ROSEVILLE, ILL.

Let m represent the number of points on a unit of surface, then is

$\pi mr^2 =$ the number of points on a circle, radius r, and

$\pi^3m^3r^6 =$ " " of positions of thee points on circle, radius r;

$2\pi r\sqrt{m} =$ " " of points on the circumference $2\pi r$

$=$ " " of positions of two points on circumf. and at a constant distance from each other.

$4\pi^2r^2m =$ " " of positions of two points on the circumf.;

$8\pi^3r^3m\sqrt{m} =$ " " " " of three points on circumf.

$\pi m(r-x)^2 =$ " " " " wholly in the given circle, which a circle of radius x, can have. Hence

$8\pi^4m^2\sqrt{m}.(r-x)^2x^3 =$ the number of positions which three points can have on the circle r and at the same time on the circumference of the within circle of radius x. Therefore

$$\int_0^r 8\pi^4m^2\sqrt{m}.(r-x)^2x^3\sqrt{m}.dx = \tfrac{2}{15}\pi^4m^3r^6,$$

$=$ the number of positions of three points, on circle r, through each of which positions a within circle may be passed. Therefore

$$\frac{\frac{2}{15}\pi^4m^3r^6}{\pi^3m^3r^6} = \frac{2\pi}{15} = \text{the required probability.}$$

[Mr. Seitz has furnished a very elaborate solation of this problem and obtains for the required probability $p = \frac{2}{5}$. We have not examined, critically, either solution, and cannot say whether the discrepancy in the results arises from error in reduction or error in logic. If it shall be found that there is error in the logic of the foregoing solution, we will publish Mr. Seitz's solution in No. 2.—Ed.]

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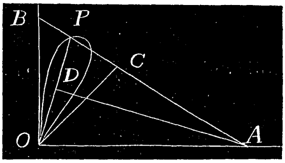
287. "A ladder, 40 feet long, stands upright against a vertical wall, on a horizontal plane. A boy ascends the ladder, at a uniform rate, and at the same time the foot of the ladder slides out horizontally at the same uniform rate. How long is the boy's path?"

SOLUTION BY GEORGE EASTWOOD, SAXONVILLE, MASS.



In the annexed diagram, let $AB = a$ represent the ladder and P the position of the boy at any time. Join OP and draw AD perpendicutar to it.

Put $OP = r$ and $\angle BOP = \theta$. Then, since AD bisects OP (because $AO = AP$, as per quest.) the $\angle OAD = \angle BOP = \theta$; $\therefore AO = a\cos 2\theta$, and $OP = r = 2OD$; $\therefore r = 2a\sin\theta\cos 2\theta$, the polar equation of the curve.

To find its length: We have

$$\frac{ds}{d\theta} = \left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{1}{2}}$$
$$= 2a(1-12\sin^2\theta+44\sin^4\theta-32\sin^6\theta)^{\frac{1}{2}}$$
$$= 2a(1-6\sin^2\theta+4\sin^4\theta+8\sin^6\theta+\ldots)$$

Whence $s = 2a\int(1-6\sin^2\theta+4\sin^4\theta+8\sin^6\theta+\ldots)d\theta$

$$= 2a[2\theta+\ldots-\cos(\sin\theta+\tfrac{8}{3}\sin^3\theta+\tfrac{4}{3}\sin^5\theta+\ldots)],$$

in which the limits of θ are 0 and 45°.

The area of the curve $= .0594a^2$.

[Prof. Hyde solves this problem also by quaternions and obtains for the Scalar equation to the curve

$$\rho^4 - 2l\rho^2 Sa\rho - 4lS^3a\rho = 0;$$

from which he deduces the Cartesian equation,

$$(x^2+y^2)^2-2lx(x^2+y^2)+4lx^3 = 0.$$

Mr. Harvill finds the rectangular equation

$(y^2+x^2)^2 = 2ax(y^2-x^2)$, and the polar equation

$r = 2a\sqrt{2}.(\sin\theta\cos^2\theta-\sin^2\theta\cos\theta)$.—Ed.]

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288. No solution of this problem has been received.